

An Improved Energy-Area Inequality for Harmonic Maps Using Image Curvature

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Abstract

An ODE variational calculation shows that an image principle curvature ratio factor, $\left(\sqrt{\frac{\rho_1}{\rho_2}} + \sqrt{\frac{\rho_2}{\rho_1}}\right)$, can raise the lower bound, $2(\text{Image Area})$, on energy of a harmonic map of a surface into \mathbb{R}^n . In certain situations, including all radially symmetry harmonic maps, equality is achieved.

1 Introduction

The energy inequality for harmonic maps of surfaces into \mathbb{R}^n , achieves equality if and only if the map is conformal and the image has zero mean curvature [J][ES][EF][EL1][EL2]. We extend this connection between principle curvatures and conformality of the harmonic map to non-minimal surfaces. Note principle curvatures in \mathbb{R}^n can be defined using the vector valued second fundamental form. As the ratio of principle curvatures deviates from 1, the minimal surface case, the map becomes higher energy and less conformal. This enables the lower bound on energy to be raised based on a principle curvature ratio term.

The results hinge on an ODE variational calculation using second order derivatives of maps that are approximated to second degree (4) using principle curvature information. This requires well known regularity results, e.g.: [L] (thm 2.1.11), about harmonic maps of surfaces into \mathbb{R}^n to be sure that (4) is defined almost everywhere on images of harmonic maps.

2 Physical intuition

We say that as the image deviates more from being a minimal surface, more energy is required of the harmonic map. A physical interpretation is to consider an elastic sheet with low curvature in one direction and high curvature in the other. To maintain equilibrium, the tension in the sheet in the low curvature direction must be much greater than in the other direction, thus contributing more to energy.

3 The Inequality

Theorem 1 *If h is a degree 1 harmonic map from a smooth compact surface into \mathbb{R}^n , ρ_1 and ρ_2 are principle curvatures of the image of h , then*

$$\text{Energy} \geq \iint_{\text{image}} \left(\sqrt{\frac{\rho_1}{\rho_2}} + \sqrt{\frac{\rho_2}{\rho_1}} \right) di dj \geq 2(\text{area of image}) \quad (1)$$

whenever the integral makes sense on the image, taking $0/0=1$. Also for $a>0$, $a/0=\infty$.

Furthermore when the pull back to the domain of the directions of principle curvatures are defined, we can say that equality is achieved on the left hand side if and only if the pull back of the directions of principle curvatures are orthogonal in the domain. This occurs for the radially symmetric case.

Remark: When $\rho_1=\rho_2$, the right hand inequality becomes equality. The left hand inequality depends upon a version of the conformality of the map. When maps are conformal and the image is a minimal surface, the energy is twice the area of the image.

Proof of Theorem 1

Consider a local region of the image of a degree 1 harmonic map h in \mathbb{R}^3 , where it has principle curvatures ρ_1 and ρ_2 . This can be generalized to \mathbb{R}^n using the vector valued second fundamental form, so we shall continue the discussion only in \mathbb{R}^3 . Take intrinsic coordinates in the image i and j which are locally orthonormal and parallel to the directions of principal curvatures, and having origin at $h(p)$. Their pullbacks in the domain are directions r and s , having origin at p . Following the set up in figure 1, define local orthonormal coordinates u, v , on the domain. Finally the range has orthonormal coordinates X , and Y , tangential to i and j at $h(p)$ and Z , parallel to principal curvature radii at $h(p)$. See figure 1.

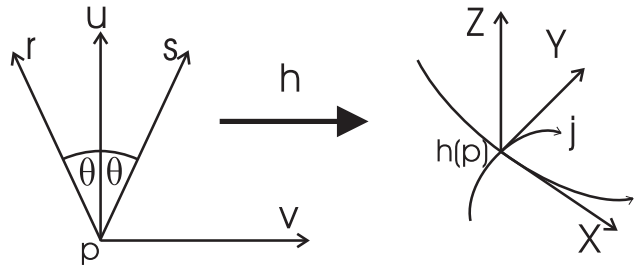


Figure 1: Coordinate systems on domain and range

Let the choice of u, v, s and t be such that with linearization.

$$\begin{aligned} r &= u \cos(\theta) - v \sin(\theta) \\ s &= u \cos(\theta) + v \sin(\theta) \\ i_r(0,0) &= X_r(0,0) = a \geq 0, j_r(0,0) = Y_r(0,0) = 0 \end{aligned}$$

$$\begin{aligned}
i_s(0,0) &= Xs(0,0) = 0, \quad j_s(0,0) = Ys(0,0) = b \geq 0 \\
Z_r(0,0) &= 0, \quad Zs(0,0) = 0, \\
\begin{bmatrix} i \\ j \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \\
\begin{bmatrix} r \\ s \end{bmatrix} &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
X_u^2(0,0) + X_v^2(0,0) &= X_r^2(0,0) = i_r^2(0,0) \text{ and} \\
Y_u^2(0,0) + Y_v^2(0,0) &= Y_s^2(0,0) = j_s^2(0,0)
\end{aligned}$$

Now the energy of the map is

$$\begin{aligned}
& \iint_{\text{domain}} X_u^2 + X_v^2 + Y_u^2 + Y_v^2 + Z_u^2 + Z_v^2 dudv \quad (2) \\
&= \iint_{\text{image}} \frac{1}{|i_r j_s \sin(2\theta)|} (i_r^2 + j_s^2) didj \\
&= \iint_{\text{image}} \frac{1}{|ab \sin(2\theta)|} (a^2 + b^2) didj \\
&= \iint_{\text{image}} \frac{1}{|\sin(2\theta)|} \left(\frac{a}{b} + \frac{b}{a} \right) didj \quad (3)
\end{aligned}$$

We will now find the ratio a/b in terms of the ratio of principal curvatures on the image using a variational calculation. Let X , Y , and Z be orthonormal coordinates on the range as in figure 1. So we can write formulae for the image coordinates, X , Y and Z in terms of image coordinates i and j and domain coordinates r and s . The principle curvatures are ρ_1 and ρ_2 . To first degree $i = ar$ and $j = bs$ so $\rho_1 ar$ and $\rho_2 bs$ are linear approximations to the angle of rotation of the image tangent plane, in directions of principle curvatures, with respect to the tangent plane at the origin. This gives us the Z coordinate, positive in the direction of the radius of curvature ρ_1 , based on a second degree approximation:

$$Z(r, s) = \frac{1}{\rho_1} (1 - \cos(\rho_1 ar)) - \frac{1}{\rho_2} (1 - \cos(\rho_2 bs)) + o(r^3, r^2 s, r s^2, s^3) \quad (4)$$

Now we place a smooth deformation field $\tau(X, Y)$ with compact support in the Z direction, and let it act with constant velocity, with time t . Now we calculate the first variation of energy under the deformation.

$$\frac{d}{dt}Energy = \frac{d}{dt} \left(\iint_{image} \tilde{Z}_u^2 + \tilde{Z}_v^2 dudv \right) = \iint_{image} \frac{1}{|\sin(2\theta)|} \frac{d}{dt} \left(\tilde{Z}_r^2 + \tilde{Z}_s^2 \right) drds$$

$$\begin{aligned} \tilde{Z}(u, v, t) &= Z(u, v) + t\tau(X(u, v), Y(u, v)) \\ \tilde{Z}_u(u, v, t) &= Z_u(u, v) + t\tau_u(X(u, v), Y(u, v)) \\ \tilde{Z}_u^2(u, v, t) &= Z_u^2(u, v) + 2tZ_u(u, v)\tau_u(X(u, v), Y(u, v)) + t^2\tau_u^2(X(u, v), Y(u, v)) \\ \frac{d}{dt} \left(\tilde{Z}_u^2(u, v, t) \right) &= 2Z_u(u, v)\tau_u(X(u, v), Y(u, v)) = 2Z_u\tau_u \end{aligned}$$

Now we can write down the first variation of energy and simplify as follows:

$$\frac{d}{dt}Energy = 2 \iint Z_u\tau_u + Z_v\tau_v dudv$$

integrating by parts

$$= -2 \iint (Z_{uu} + Z_{vv}) \tau dudv$$

Note that τ is smooth and bounded with compact support. This yields the standard condition for a harmonic map that the coordinate functions are harmonic:

$$Z_{uu} + Z_{vv} = 0 \quad (5)$$

Now using $Z_{rs} = o(r, s)$ and $Z_s = o(r^2, rs, s^2)$, $Z_r = o(r^2, rs, s^2)$ and from differentiating (4), we can obtain:

$$\begin{aligned} Z_u &= r_u Z_r + s_u Z_s + o(r^2, rs, s^2) \\ Z_{uu} &= \frac{\partial}{\partial u} (r_u Z_r + s_u Z_s) + o(r, s) \\ &= r_{uu} Z_r + r_u Z_{ru} + s_{uu} Z_s + s_u Z_{su} + o(r, s) \\ Z_{uu}(0, 0) &= (r_u Z_{ru} + s_u Z_{su}) \\ Z_{ru} &= r_u Z_{rr} + s_u Z_{rs} + o(r, s) \\ Z_{ru}(0, 0) &= r_u Z_{rr} \end{aligned}$$

Now we can ignore higher order terms, giving:

$$Z_{uu} = r_u^2 Z_{rr} + s_u^2 Z_{ss} \quad (6)$$

$$Z_{vv} = r_v^2 Z_{rr} + s_v^2 Z_{ss} \quad (7)$$

Now we are evaluating r_u , r_v , s_u and s_v at $(0, 0)$. Therefore we can use the linearization, as derivatives of quadratic terms on the Taylor expansions of r and s will be zero at $(0, 0)$. Also note that in the Taylor expansion θ is a constant. So we can use:

$$\begin{aligned}
r_u &= \cos(\theta) + o(r, s), & -r_v &= \sin(\theta) + o(r, s), \\
s_u &= \sin(\theta) + o(r, s), & s_v &= \cos(\theta) + o(r, s), \\
r_u^2(0, 0) + r_v^2(0, 0) &= s_u^2(0, 0) + s_v^2(0, 0) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1
\end{aligned}$$

substituting into (6) and (7) gives:

$$Z_{uu}(0, 0) + Z_{vv}(0, 0) = Z_{rr}(0, 0) + Z_{ss}(0, 0)$$

This point-wise calculation was for an arbitrary regular point, (u_0, v_0) , where we set the local origin $(0, 0)$ in our local coordinates. We can repeat this for all regular points of the map. That is almost everywhere. We can now say for the map to be energy stationary we obtain the condition in terms of r and s :

$$Z_{rr} + Z_{ss} = 0 \tag{8}$$

Applying (4) to (8) gives:

$$\begin{aligned}
Z(r, s) &= \frac{1}{\rho_1} (1 - \cos(\rho_1 ar)) - \frac{1}{\rho_2} (1 - \cos(\rho_2 bs)) + o(r^3, r^2s, rs^2, s^3) \\
Z_r &= \frac{\rho_1 a}{\rho_1} (\sin(\rho_1 ar)) + o(r^2, rs, s^2) \\
Z_{rr} &= \rho_1 a^2 (\cos(\rho_1 ar)) + o(r, s) \\
Z_{ss} &= -\rho_2 b^2 (\cos(\rho_2 bs)) + o(r, s) \\
\rho_1 a^2 (\cos(\rho_1 ar)) - \rho_2 b^2 (\cos(\rho_2 bs)) + o(r, s) &= 0
\end{aligned}$$

Now taking r and s arbitrarily small by controlling the support of τ we can equate the constant terms resulting from the Taylor expansions to obtain the relationship:

$$\rho_1 a^2 - \rho_2 b^2 = 0 \Leftrightarrow \frac{a}{b} = \sqrt{\frac{\rho_2}{\rho_1}} \tag{9}$$

This gives us a lower bound on the energy in terms of the image and principal curvatures on the image using (3) and (9), when the integrals make sense:

$$Energy = \iint_{image} \frac{1}{|\sin(2\theta)|} \left(\sqrt{\frac{\rho_1}{\rho_2}} + \sqrt{\frac{\rho_2}{\rho_1}} \right) didj \tag{10}$$

$$\geq \iint_{image} \left(\sqrt{\frac{\rho_1}{\rho_2}} + \sqrt{\frac{\rho_2}{\rho_1}} \right) didj \tag{11}$$

$$\geq 2(area \text{ of image}) \tag{12}$$

■

Now we can see that the condition $a = b$ corresponds to the principal curvatures being equal, hence the surface being minimal. Also when $a = b$ and the map is conformal, $\sin(2\theta) \equiv 1$ in (10). So we conclude that energy equals twice

the area of the image when the image is minimal and the map is conformal as (10) and (11) become equalities.

Note the integral is defined when the quantity $\sqrt{\frac{\rho_1}{\rho_2}} + \sqrt{\frac{\rho_2}{\rho_1}}$ is defined, i.e.: only for negatively curved surfaces, not ruled or planar. Positive curvature does not arise in images of harmonic maps into \mathbb{R}^n . In the planar case, $\rho_1 = \rho_2 = 0$ use $\sqrt{\frac{\rho_1}{\rho_2}} = 1$. Note that this corresponds to $a = b$ in (3) and (11) and (12) become equality.

In the radially symmetric case, $h : (r, \phi) \rightarrow (R, \Phi)$. R and Φ are only functions of one variable, r and ϕ respectively. Therefore in (10) $\sin(2\theta) \equiv 1$. Thus the inequality (11) becomes an equality. This means that energy is completely determined by the image, its area and curvatures.

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